Received: *June 27, 2001*, Accepted: *August 03, 2001*

String field theory, non-commutative Chern-Simons theory and Lie algebra cohomology

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Abstract: Motivated by non-commutative Chern-Simons theory, we construct an infinite class of field theories that satisfy the axioms of Witten's string field theory. These constructions have no propagating open string degrees of freedom. We demonstrate the existence of non-trivial classical solutions. We find Wilson loop-like observables in these examples.

Keywords: Chern-Simons Theories, D-branes, Non-Commutative Geometry.

JHEP08(2001)008

Contents

1. Introduction

String field theory [1] has enjoyed a renaissance of late [2]. It appears to provide a framework for concrete calculations to further explore the ramifications of Sen's ideas on the physical consequences and the interpretation of tachyon condensation [3] in open string theory.

Some recent work has focussed on string field theory with a modified BRST operator Q which is pure ghost and linear in ghost fields [4]–[8]. Such an operator naturally has no non-trivial physical open string states, and as such provides a putative explanation for the vanishing of open strings in the stable vacuum of the bosonic open string theory. With such a pure ghost Q solutions of the string field theory equations of motion can be taken to have a factorized form

$$\Psi \equiv \Psi_g \otimes \Psi_m : Q\Psi_g + \Psi_g * \Psi_g = 0\,, \qquad \text{and} \qquad \Psi_m * \Psi_m = \Psi_m\,. \tag{1.1}$$

An interesting aspect of these studies is that the BRST operator is linear in ghost fields, and the integration operation of the string field theory is left unchanged. We present in this work an infinite set of solutions of the axioms of string field theory:

$$\begin{aligned} Q^2 &= 0\,, \\ \int Q\Psi &= 0\,, \\ Q(\Psi * \Phi) &= Q\Psi * \Phi + (-)^{|\Psi|}\Psi * Q\Phi\,, \\ \int \Psi * \Phi &= (-)^{|\Psi||\Phi|}\int \Phi * \Psi \end{aligned} \tag{1.2}$$

which have the desired properties:

1. no physical open string states,
2. closed string observables, and
3. non-trivial classical solutions.

These examples are associated with the cohomology of Lie algebras. The only non-obvious aspect of our construction is the definition of an integration operation on the exterior algebra associated with the adjoint representation module of the Lie algebra.

These constructions arise as non-commutative analogues of Chern-Simons theory [9]. Why might this be of interest? The action of Chern-Simons theory is formally very similar to the action of Witten's string field theory, which has been shown to simplify greatly in the limit of strong noncommutativity. Chern-Simons theory on commutative manifolds has correlation functions of Wilson lines as observables — these are the knot invariants associated with statistical mechanics models on plane projections of knots in three embedding dimensions. The natural observables in Witten's string field theory seem to be quite different. From the perspective of gauge theories on non-commutative spaces, it is actually difficult to understand what to make of Wilson line expectation values since such Wilson lines on some examples of non-commutative manifolds appear to be gauge invariant only if they are integrated over the manifold. Obviously, such an integration renders a knot invariant interpretation of Wilson line correlation functions difficult. Thus it seems likely that observables of non-commutative Chern-Simons gauge theory might be much more interesting from the perspective of models of Witten's string field theory than commutative Chern-Simons theory.

This paper is organized as follows: in section 2, we consider the problem of defining an odd-dimensional non-commutative manifold. We carry this out explicitly for a three-dimensional case. In section 3 we construct the Chern-Simons theory associated with this non-commutative threefold. A general framework for finding infinite classes of examples is then immediate. In the concluding section, we find non-trivial closed string-like observables for the gauge theory, after explaining why there are no open string states in the spectrum.

2. Non-commutativity in three dimensions

What is a natural notion of a odd-dimensional non-commutative manifold? If we start with a non-commutative even-dimensional manifold associated with a deformation quantization of a Poisson structure, it would appear that there are two possible ways in which an odd-dimensional non-commutative manifold might be defined, either as a contact submanifold associated with a choice of a level set for a hamiltonian

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or as a non-commutative contact manifold in one higher dimension associated with a time-dependent hamiltonian. We consider the former construction in order to define a non-commutative threefold.

We shall start with an explicit example and then abstract from the discussion to arrive at a conclusion that is very simple to state: non-commutative Chern-Simons gauge theory is formulated along the lines of Witten's string field theory with Lie algebra cohomology playing the rôle of Q.

If P is a symplectic manifold with a hamiltonian function H and $\Sigma(E)$ is a regular energy surface, then the "restriction" to $\Sigma(E)$ is a contact manifold. The two-form ω is just the pullback of the symplectic form of P. Exact contact manifolds locally have a one-form that takes the form $\theta = dw + p_i dq^i$, where (w, p_i, q^i) are the local coordinates.

We can obviously define a non-commutative structure on any symplectic 4-manifold. Work with the simplest 4d symplectic structure $\omega = dp_1 dq^1 + dp_2 dq^2$. The non-commutative structure defined by this ω just factorizes into a tensor product of two isomorphic copies of the algebra associated with the non-commutative plane — we will denote this algebra by M. Now we define the algebraic analogue of restricting to $\Sigma(E)$. The first thing we need is to pick an energy function. To find a good algebra to associate to constant energy surfaces in this phase space, we define first

$$I_E = \{f * (H - E) | f \in M\} . \tag{2.1}$$

This is the left ideal of functions that vanish on the constant energy surface. By definition if $g \in I_E$ then $h * g \in I_E$ by the associativity of the $*$ product. We now want to find a subalgebra N_E of M such that I_E is a two-sided ideal in N_E. It will then follow that N_E/I_E is an algebra—M/I is not an algebra in general unless I is a two-sided ideal in M. Define

$$N_E = \{f \in M | (H - E) * f \in I_E\} . \tag{2.2}$$

What this means explicitly is that for any element f in N_E there is an element g in M such that

$$(H - E) * f = g * (H - E) . \tag{2.3}$$

Thus I_E is a two-sided ideal in N_E. The quotient algebra $\mathcal{M} \equiv N_E/I_E$ is the non-commutative analogue of the algebra of functions on a contact manifold.

For the phase space of two decoupled identical harmonic oscillators (with creation and annihilation operators $a_i^\dagger, a_i;\ i = 1, 2$, and hamiltonian $H = a_1^\dagger a_1 + a_2^\dagger a_2 + 1$), for example, it is easy to work out what operators are in N_E. These are operators that do not change the total occupation number of the two decoupled harmonic oscillators. So the quotient algebra $\mathcal{M}$ is generated by $\zeta \equiv a_1^\dagger a_2$, $\zeta^\dagger$ and $h \equiv a_1^\dagger a_1 - a_2^\dagger a_2$. These operators satisfy the SU(2) commutation relations, not surprising since we expect the energy surface to be topologically an S^3, namely the surface $H = (x_1^2 + p_1^2 +$

$x_2^2 + p_2^2)/2 = E$. For appropriate values of the energy, the quotient algebra has the well-known finite-dimensional representations:

$$T_\zeta = \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad T_{\zeta^\dagger} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

$$T_h = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}, \qquad [T_a, T_b] = f_{ab}^c T_c \,. \tag{2.4}$$

The Casimir element $C = \zeta\zeta^\dagger + \zeta^\dagger\zeta + h^2/2$ generates the centre of the enveloping algebra.

Now we turn to the algebra of forms over $\mathcal{M}$. We introduce operators c^a and b_a where $a \in \{\zeta, \zeta^\dagger, h\}$, such that $\{c^a, b_d\} = \delta_d^a$ with $\{b, b\} = 0 = \{c, c\}$. Then the operator

$$Q = c^a T_a - \frac{1}{2} f_{de}^g c^d c^e b_g \tag{2.5}$$

squares to zero and is the well-known operator computing Lie algebra cohomology with values in the representation module specified by the matrices T. Note that $Q^2 = 0$ for any Lie algebra, the only properties of f_{bc}^a used in proving this are $f_{bc}^a = f_{cb}^a$ and the Jacobi identity. Also note that

$$\{Q, b_a\} = T_a - f_{ae}^g c^e b_g \equiv \mathcal{T}_a \,. \tag{2.6}$$

$\mathcal{T}_a$ consists of two pieces, the first rotates elements of the algebra in the representation of the T's and the second rotates the ghosts in the adjoint representation.

The interpretation here is that Q is the non-commutative exterior derivative, with the algebra of forms given by $\mathcal{M}[c^a]$, the (graded-)commutative polynomial algebra generated by c^a with coefficients in $\mathcal{M}$. With the definition $[b_a, x] = 0 = [c^a, x]$ for any $x \in \mathcal{M}$ we see that Q maps forms to forms, raising the degree by one. Notice that $\{Q, c^a\} + \frac{1}{2} f_{bc}^a c^b c^c = 0$, so the c^a form an orthonormal basis and $\frac{1}{2} f_{bc}^a c^b$ is the spin connection.

3. Non-commutative Chern-Simons theory

To define a gauge theory over this algebra we need to pick a projective $\mathcal{M}$-module and define the Chern-Simons integrand in terms of a connection on this module. The simplest example of a module is of course $\mathcal{M}$ itself, which defines for us a U(1) gauge theory. The integrand is easy enough to write down formally but it isn't completely obvious what integration is appropriate. We define

$$S_{\rm ncACS} = {\rm Tr} \int_c \left[\Psi\{Q,\Psi\} + \frac{2}{3}\Psi^3 \right], \tag{3.1}$$

where $\Psi = c^a A_a$ is a 1-form with components A_a. We expect that there should be some natural notion of integration for a three-form, and we expect that the trace over matrix indices in the representation T should be the analogue of integration over the threefold.

Recall that the non-abelian commutative Chern-Simons action takes the form

$$S_{\rm CS} = \int \epsilon^{ijk} \left(\delta_{ab} A_i^a \partial_j A_k^b + \frac{2}{3} f_{abc} A_i^a A_j^b A_k^c \right), \tag{3.2}$$

where $A_i dx^i$ is the gauge potential. In that context the trace over the adjoint representation provides the factors δ_{ab} and f_{abc}. In our case, that of abelian but non-commutative gauge theory, the matrix trace takes the place of $\int \epsilon^{ijk}$. We therefore have to figure out what to do with the cubic term in c^a that takes the place of the three-form. Given the constraint that the integration $\int_c c^a c^b c^d$ must be completely antisymmetric, the only natural assignment is to define

$$\int_c c^a c^b c^d = f^{abd}, \tag{3.3}$$

where f are the structure constants of the Lie algebra of SU(2).

Thus far we have restricted to the case of SU(2) which arises naturally from our contact manifold construction. It turns out that there is nothing specific to SU(2) in our construction, so henceforth we shall work with an arbitrary semi-simple Lie algebra $\mathcal{G}$ and an arbitrary (possibly reducible) representation T acting on a vector space V. In particular, $\mathcal{M}$ will be the complete operator algebra acting on V, not just the enveloping algebra of $\mathcal{G}$. $\mathcal{M}$ coincides with the enveloping algebra if and only if T is an irreducible representation.

To verify that this is a suitably gauge-invariant action, that the gauge invariance is the usual non-abelian gauge invariance, and that the axioms of string field theory are satisfied, we first note that the action of Q on a field Ψ of ghost number $|\Psi|$ is $Q\Psi - (-1)^{|\Psi|}\Psi Q$. Furthermore,

$${\rm Tr} \int_c [Q,\chi] = 0 \tag{3.4}$$

for any χ. In fact, only $|\chi| = 2$ can contribute so the only choice of ϵ that can give a non-vanishing contribution given our definition of $\int_c$ is

$$\chi = \chi_{ab} c^a c^b \tag{3.5}$$

where χ_{ab} is a matrix in $\mathcal{M}$ for every choice of the indices a and b. Clearly $\chi_{ab} = -\chi_{ba}$.

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The gauge transformations take the form

$$\delta\Psi \equiv [Q + \Psi, \epsilon] \,, \tag{3.6}$$

where ϵ is a ghost number zero field. Any ϵ not satisfying this would of course trivially leave the action invariant.

We can make this completely explicit in terms of component fields. It can be directly verified that

$$S_{\mathrm{ncACS}} = f^{abc}\,\mathrm{Tr}\left(A_a\,[T_b, A_c] - \frac{1}{2} f^{bcd} A_a A_d + \frac{2}{3} A_a A_b A_c\right) \tag{3.7}$$

is invariant under

$$\delta_\epsilon A_a \equiv [\mathcal{T}_a + A_a, \epsilon] = [T_a + A_a, \epsilon] \,, \tag{3.8}$$

for any ϵ in $\mathcal{M}$. We have therefore arrived at a definition of a non-commutative Chern-Simons action which satisfies all the axioms of Witten's string field theory. Note the presence of the spin connection coupling in S_{ncACS}.

A natural question is: what does this general non-commutative abelian Chern-Simons action have to do with threefolds? The answer requires first asking what characteristics one wants in a threefold. In the non-commutative setting, an abstract threefold is something that can be paired with three-forms to give a number. So the logic is: given an algebra $\mathcal{M}$ one can define an exterior differential calculus over the algebra. The next step involves the analogue of integrating over a threefold, and that corresponds to finding a character of this algebra. This is what the integration operation given above defines. It would be interesting to find explicit energy functions on symplectic manifolds such that the algebras associated with the non-commutative analogues of contact manifolds are explicitly generated by semi-simple Lie algebra representations. An extension of our construction to the case of superalgebras may also be of interest. In that case it would be natural for $\mathcal{M}$ to contain anticommuting elements as well which would lead to more component fields in Ψ.

4. Solutions and observables

We turn now to the issue of classical solutions and observables [6, 7].

4.1 Classical solutions

The first observation is that the equation of motion $(Q + \Psi)^2 = 0$ may be written in components as

$$[T_b + A_b, T_c + A_c]\, f^{abc} = f^{abc} f^{dbc}\,(T_d + A_d) = C(G)\,(T_a + A_a) \,, \tag{4.1}$$

which essentially defines Lie algebras. At a solution of this equation, the value of the action is

$$S_{\mathrm{cl}} = -\frac{1}{6}\,\mathrm{Tr}(2T_a A_a + A_a A_a) C(G) \,. \tag{4.2}$$

Are there any non-trivial solutions of this equation of motion? If T is a finite-dimensional reducible representation then in fact there is a very simple set of non-trivial solutions. If P is the projector onto a given sub-representation of T, then $A = -PTP$ is a solution of the classical equation of motion such that $T + A = (1-P)T = T(1-P)$.

Considerably more interesting solutions are possible in the infinite-dimensional case. If $S^\dagger$ is an operator satisfying $S^\dagger S = 1$ and $SS^\dagger = 1-P$ where P is a projection, we can construct new solutions by setting $T + A = STS^\dagger$ since we have then

$$S[T_b, T_c]S^\dagger f^{abc} = C(G) S T_a S^\dagger \,. \tag{4.3}$$

S is an example of a partial isometry, so there is a direct connection to a charge in operator K theory corresponding to such solutions [10]. Let us compute the value of the classical action for these solutions. We find

$$S_{\rm cl} = -\frac{1}{6}\,\mathrm{Tr}(ST_aT_aS^\dagger - T_aT_a) = -\frac{1}{6}\,\mathrm{Tr}((1-P)ST_aT_aS^\dagger - T_aT_a)\,. \tag{4.4}$$

This expression shows how the value of the action is related to the rank of the projection P, and also indicates that care is needed in evaluating such infinite-dimensional traces. It is interesting to note that the quantum theory has off-diagonal fluctuations in A which couple different irreducible sub-representations in any given solution of these classical equations, which heuristically are analogous to closed string couplings generated by open string loops.

4.2 Physical states

In the BRST formalism we want to compute the cohomology of $Q + \Psi$. There are several ways to compute this cohomology, which turns out to be the cohomology of the Lie algebra, $\mathcal{G}$. Recall the definition of Lie algebra cohomology: if V is the representation space of a representation T of $\mathcal{G}$, an n-dimensional V-cochain is a skew-symmetric multilinear mapping: $\mathcal{G}^{\times n} \to V$. The coboundary operator s is defined by its action on n-cochains:

$$\begin{aligned}(s\omega)\left(X_1,\ldots,X_{n+1}\right) = &\sum_{i=1}^{n+1}(-)^{i+1}T\left(X_i\right)\omega\left(X_1,\ldots,\hat{X}_i,\ldots,X_{n+1}\right) + \\ &+ \sum_{j,k=1;j<k}^{n+1}(-)^{j+k}\omega\left(\left[X_j,X_k\right],X_1,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_{n+1}\right),\end{aligned}$$

where X_i are elements of $\mathcal{G}$, and $\hat{}$ denotes omission. For example, $s\omega(X_1, X_2) = T(X_1)\omega(X_2) - T(X_2)\omega(X_1) - \omega([X_1, X_2)$. Using the fact that T is a representation, it is trivial to verify that $s^2 = 0$. Thus cohomology groups for s associated with T with values in V are readily defined as the vector space of cocycles (cochains annihilated by s) modulo the coboundaries (cochains of the form $s\omega$). This s operator is related

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to Q by a factor of $(n+1)$ for an n-cochain so the cohomology groups of Q and s coincide. For example, $H^0_T(\mathcal{G}, V) = V^{\mathcal{G}}$, where $V^{\mathcal{G}}$ is the set of vectors left invariant by the action of $\mathcal{G}$. This result is obtained by noting that zero-cochains, which are just vectors in V, are zero-cocycles if and only if $sv(X) = T(X)v = 0$ for all X in $\mathcal{G}$. In particular, for irreducible representations there are no invariant vectors. It is also easy to show that for semisimple $\mathcal{G}$, $H^1_T(\mathcal{G}, V) = 0$: a 1-cocycle is a 1-cochain that satisfies

$$s\omega\left(X_1, X_2\right) = T\left(X_1\right)\omega\left(X_2\right) - T\left(X_2\right)\omega\left(X_1\right) - \omega\left(\left[X_1, X_2\right]\right) = 0 \qquad (4.5)$$

for all X_1, X_2 in $\mathcal{G}$. On the other hand, a 1-cocycle ϖ is a coboundary if $\varpi(X) = T(X)v$ for all X in $\mathcal{G}$ and some v in V. Define linear maps $h(Y)$ (for Y in $\mathcal{G}$) which take 1-cocycles to 1-coboundaries by $(h(Y)\omega)(X) = T(Y)\omega(X) - \omega([Y, X]) = T(X)\omega(Y)$. Now suppose there is a 1-cocycle ω which is annihilated by $h(Y)$ for all Y in $\mathcal{G}$. This means $T(X)\omega(Y) = 0$ for all X, Y in $\mathcal{G}$, but then using the definition of a 1-cocycle, it follows that $\omega([X, Y]) = 0$ for all X, Y in $\mathcal{G}$. If $\mathcal{G}$ is semisimple, $[\mathcal{G}, \mathcal{G}] = \mathcal{G}$, so ω vanishes on all of $\mathcal{G}$. This is what we wished to demonstrate. With a little more algebra one can also show $H^2_T(\mathcal{G}, V) = 0$.

Applying this cohomology computation to our theory, we see that there are no observables at ghost number 1 or 2 for any representation T. At ghost number 0 the only observables are invariants of the representation T. So there would appear to be no non-trivial perturbative open string states in our theory for semisimple Lie algebras.

4.3 Observables

What are the observables in this theory? The covariant derivative operator

$$\mathcal{D} = Q + \Psi$$

transforms homogeneously under gauge transformations, $\mathcal{D} \to \mathcal{D} + [\mathcal{D}, \epsilon]$. In components we can define

$$\mathcal{D}_a \equiv \{b_a, Q + \Psi\} = \mathcal{T}_a + A_a\,, \qquad (4.6)$$

or separate out the matter component to define

$$D_a \equiv T_a + A_a\,, \qquad (4.7)$$

Both $\mathcal{D}_a$ and D_a transfrom covariantly under gauge transformations and therefore any product of the form

$$\mathcal{O} = \mathrm{Tr}\prod_i \left(\mathcal{T}_{a_i} + A_{a_i}\right), \qquad \text{or} \qquad O = \mathrm{Tr}\prod_i \left(T_{a_i} + A_{a_i}\right) \qquad (4.8)$$

is gauge-invariant. A different basis for this set of observables is

$$W\left[\lambda_i\right] \equiv \mathrm{Tr}\prod_i \exp\left(\lambda_i\left(T_{a_i} + A_{a_i}\right)\right) \qquad (4.9)$$

which are like the Wilson lines studied in non-commutative gauge theories.

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We can make this analogy stronger by recalling eq. (4.6) and noting the anit-commutation with b_a is differentiation with respect to c_a. However, anticommuting differentiation is the same as integration so we can alternatively think of an abstract set of non-commutative 1-cycles, much as we thought of integration on a non-commutative threefold, indexed by the generators of $\mathcal{G}$. Define

$$\oint_b c^a = \delta_b^a \,. \tag{4.10}$$

Then

$$\oint_b (Q + \Psi) = \mathcal{T}_b + A_b \,. \tag{4.11}$$

This is written as an operator equation. We can now write

$$\begin{aligned} W[\lambda_i] &= \mathrm{Tr} \prod_i \exp\left(\lambda_i \left(T_{a_i} + A_{a_i}\right)\right) \\ &= \mathrm{Tr} \prod_i \exp\left(\lambda_i \oint_{a_i} (Q+\Psi)\right) \\ &= \mathrm{Tr}\, P \exp\left(\oint_{\sum \lambda_i a_i} (Q+\Psi)\right), \end{aligned} \tag{4.12}$$

where $\sum \lambda_i a_i$ is formally a (path ordered) 1-cycle. The ghost part of the group element factors out. Thus in the vacuum ($\Psi = 0$), or more generally in the background of a classical solution, where $(Q + \Psi)^2 = 0$, $W[\lambda_i]$ is a product of group elements in the representation associated with $Q + \Psi$, an element in the loop group. Thus the Wilson loop, $W[\lambda_i]$, associates with every path in the d-dimensional space (d= rank of the Lie algebra) given by the ordered sequence $\{\lambda_1, \lambda_2, \ldots\}$, the trace of the appropriate group transformation.

When we vary the parameters λ_i the variation in the observables amounts to an insertion of the commutator $[T_a + A_a, T_b + A_b]$ which can be written as an insertion of the equation of motion, which leads to contact terms similar to skein relations, but one also obtains an insertion of a term linear in $T + A$ since there is a torsion term in the curvature. Lastly, we note that these expressions are also reminiscent of Wilson lines in Eguchi-Kawai reduction, with the index a in the Lie algebra representing the direction.

These models may be useful in the limit of large representation size and/or large group rank, as approximations to string theory, along the lines of the matrix representation of membrane dynamics. It may be possible to generalize these models to A_∞ algebras as well. Defining a cubic action non-perturbatively is a problem that has not been solved for commutative Chern-Simons theory, so there is still some work remaining.

Acknowledgments

VP is grateful to J. Minahan for several valuable conversations during the early stages of this work and E. Witten for some helpful comments. The work of DJG was supported by the NSF under the grants PHY 99-07949 and PHY 97-22022. VP was supported by the NSF under grant PHY 98-02484. We are grateful to D. Bak for correcting an error in our manuscript.

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